Parameterized Metatheory for Continuous Markovian Logic

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***Abstract*—This paper shows that a classic metalogical framework, including all Boolean operators, can be used to support the development of a metric behavioural theory for Markov processes. Previously, only intuitionistic frameworks or frameworks without negation and logical implication have been developed to fulfill this task. The focus of this paper is on continuous Markovian logic (CML), a logic that characterizes stochastic bisimulation of Markov processes with arbitrary measurable state space and continuous-time transitions. For a parameter $\varepsilon > 0$ interpreted as observational error, we introduce an ε-parametrized metatheory for CML: we define the concepts of ε-satisfiability and ε-provability related by a sound and complete axiomatization and prove a series of "parametrized" metatheorems including decidability, weak completeness and finite model property. We also prove results regarding the relations between metalogical concepts defined for different parameters. Using this framework, we can characterize both the stochastic bisimulation relation and various observational preorders based on behavioural pseudometrics. The main contribution of this paper is proving that all these analyses can actually be done using a unified complete Boolean framework. And this extends the state of the art in this field, since the related works only propose intuitionistic contexts that limit, for instance, the use of the Boolean logical implication.**

I. Introduction

Stochastic models have successfully been used to describe the qualitative and quantitative behavior of systems in many natural and artificial domains. The problems addressed in this paper refer to the most general models of Markov processes, defined for arbitrary (analytic) state-spaces and continuous-time transitions, henceforth *continuous Markov processes* (CMPs); they subsume well-known models such as *continuous-timed Markov chains* and *labeled Markov processes* and for this reason our work can be simply instantiated for these particular models. CMPs have been initially introduced by Panangaden and Desharnais in [DP03]. In this paper, for technical reasons, we use the definition of CMPs proposed by the first two authors and Cardelli in [CLM11a], which exploits an equivalence between the definitions of *Harsanyi type spaces* [MV04] and a coalgebraic view of labelled Markov processes [dVR99] proved, for instance, by Doberkat in [Dob07].

In this paper the class of CMPs define the semantics for Continuous Markovian Logic (CML) [CLM11a], [CLM11b]. This is a multimodal logic endowed with modalities L_r, M_r for $r \in \mathbb{Q}^+$, similar to the ones used by Aumann system [Aum99], that approximates the transition rates. For instance, a process satisfies $L_r\phi$ if the rate of its transition from the initial state to a state satisfying ϕ is at least r. In [CLM11a] it has been proved that this logic characterizes the stochastic bisimulation of CMPs.

Despite the elegant theories supporting the concepts of stochastic and probabilistic bisimulations and their relation to logics [LS91], these concepts remain too strict for applications. In modelling, the values of the rates or probabilities are often approximated and consequently, one is interested to know whether two processes that differ by a small amount in real-valued parameters show similar (not necessarily identical) behaviours. In such cases, instead of a bisimulation relation, one needs a metric concept to estimate the degree of similarity of two systems in terms of their behaviours. The metric theory for Markov processes was initiated by Desharnais et al. [D+04] and greatly developed and explored by van Breugel, Worrell and others [vBW01], [vB+03]. Similar notions have been proposed in literature for less general models. It is the case of the point-wise simulation distance defined in [JS90] for discrete probabilistic systems, of the discount-future distances proposed in [AHM03], [AFS09] for weighted systems, and of the quantified similarities of timed systems studied in [MHP05].

One way of defining these behavioural distances was proposed by Kozen [Koz85] and consists in replacing the classic logical framework used to encode properties of processes with a non-classical real-valued framework that will interpret logical formulas as functional expressions mapping states to reals. In this way, we get a relaxation of the satisfiability relation which is replaced by a function that reports the "degree of satisfiability" between a Markov process and a logical property. This further induces a *behavioural pseudometric* on processes with the stochastic bisimulation as its kernel and measuring the distance between processes in terms of their behavioural similarity. Such formalisms have since been proposed for Markov systems by Desharnais, Panangaden and others [D+04], [Pan09].

It was hoped that these metrics would provide a quantitative alternative to logic, but this did not happen. One reason could originate in the fact that all this "metric reasoning" focused exclusively on the semantics of the logic while a syntactic or a metalogical counterpart did not develop until recently. Such a logical perspective on distance was proposed by the first two authors of this paper and Cardelli in [CLM11a], where it was emphasized that, in the context of a completely axiomatized logic, the semantic distance between Markovian processes implicitly induces, via Hausdorff metrics, a distance between logical properties that can be interpreted as a *measure of provability* for CML. On this line, [CLM11a] and [CLM11b] contain the open ideas of a research program that we have followed ever since. This research aims to understand the relation between the pseudometric space of Markov processes and the pseudometric space of logical formulas i.e., the relation between the measure of similarity for Markov processes and the measure of provability in a corresponding stochastic/probabilistic logic. Eventually, in [LMP12a], the first two authors in collaboration with Prakash Panangaden have identified a metric analog of Stone duality that relates the two pseudometric spaces and in [LMP12b] we have studied how convergence in the open ball topologies induced by the two pseudometrics "agree to the limit".

However, in all this work we did not succeed to clarify what is the kernel of the distance between logical formulas. We have proved in [LMP12b] that it is possible to have formulas at distance 0 that are not the logically equivalent and we have characterized this kernel for some limited fragments of CML. But the full picture has not been yet achieved. One reason for this difficulty is originated in the fact that we have no pure metalogical definition of this distance, as it is always implicitly obtained from the definition of the behavioural pseudometrics, hence it depends of the semantics. This is exactly what we try to achieve in this paper: a metalogical definition of the behavioural distance.

This paper is a step forward in the process of understanding this distance from a logical perspective. We define a parametrized metatheory for CML, where the parameter $\varepsilon \geq 0$ is interpreted as an observational error which allows us to express properties approximating the behavior of a given CMP. This metatheory consists in defining an *ε-semantics*, i.e., *ε-satisfiability relation* denoted $\models_\varepsilon$, and to develop a complete Hilbert-style axiomatization for a corresponding concept of *ε-provability*, denoted $\vdash_\varepsilon$. The classic semantics and provability relation of CML are, in this context, the 0-semantics and 0-proof theory.

This parametric metatheory allows us to transfer logical properties between various semantics defined for different parameters. For instance, we can translate 0-satisfiability into ε-satisfiability and reverse, or 0-provability into ε-provability and reverse using appropriate encodings. And exactly this feature allows one to see the behavioural pseudometrics from a logical perspective. We prove that the distance between two CMPs m_1 and m_2 can, in fact, be defined by the infimum of the set of values ε such that for any CML formula ϕ, $m_i \models \phi$ iff $m_j \models_\varepsilon \phi$, where $\{i, j\} = \{1, 2\}$.

We develop the parametric metatheory as a classic metatheory and in addition to the sound-complete axiomatization we prove a series of metatheorems such as ε-deduction theorem, ε-finite model property and ε-decidability results for CML.

The major contribution of this paper consists in the fact that this entire development respects the classic Boolean restrictions. All the attempts of realizing Kozen's idea [Koz85] and defining a quantitative version of the satisfiability relation, have faced by now serious problems related to the treatment of negation. Often in the papers treating this argument, negation is either eliminated, restricted to atomic propositions or considered in a non-Boolean context [FGK10], [DLT08]. And this restriction is an impediment for the use of classic reasoning. For instance in [DLT08], the definition of ε-satisfiability contains the following rules defined for an arbitrary Markov process m:

$$m \models_\varepsilon \neg\phi \text{ iff } m \not\models_{-\varepsilon} \phi$$
$$m \models_\varepsilon \langle a\rangle_\delta \phi \text{ iff } \theta(m, [\![\phi]\!]_\varepsilon) > \delta - \varepsilon$$

where $\theta(m, M)$ is the probability of a transition from m to a state in the set M. Observe that the rule for negation requires to transport information from "ε-semantics" to "$-\varepsilon$-semantics" and it is obviously non-Boolean. For instance, in the case $\theta(m, [\![\phi]\!]_\varepsilon) = \delta$ one can prove, using the previous rules, that for $\varepsilon > 0$ we have

$$m \models_\varepsilon \langle a\rangle_\delta \phi \wedge \neg\langle a\rangle_\delta \phi.$$

In other words, the logic is inconsistent if it is interpreted in a Boolean context.

In the light of this observation, one can see the real contribution of our paper. We show that it is possible to do the work and remain Boolean and classic to all logical levels. Of course, one can argue that an intuitionistic approach is as good for applications as the classic Boolean approach is, and we cannot argue against this. But we believe that for a deeper understanding of Markov processes and for providing a strong theoretical background for an approximation theory of Markov processes, a classic logical framework is more useful. In fact, in [LMP12a] it is identified a special Boolean algebra with operators (called *Aumann algebra*) that corresponds to CML and we proved Stone duality results between these algebras and Markov processes. This enforces our trust that the Boolean setting is the right one for studying properties of Markov processes.

To summarize, the achievements of this work are as follows.

- We develop a parametrized metatheory for continuous Markovian logic that extends the classic metatheory. The parameter can be interpreted as observational error.
- We define the concept of ε-satisfiability and identify for it an appropriate concept of ε-provability with a sound and complete Hilbert-style axiomatization.
- We prove that classic metatheorems about CML remain true in the parametric semantics. Such properties are the weak completeness, the finite model property and decidability.
- We show that this parametrized metatheory can be used to define a behavioural pseudometric, which is a distance between CMPs that characterizes the similarity of two processes from the point of view of their behaviours.

- We identify two behavioural orders that have, in the parametric semantics, similar logical interpretations to the bisimulation in the classic semantics.
- This entire development is essentially Boolean.

II. Preliminary definitions

In this section we introduce some basic notations and concepts used throughout this paper.

For arbitrary sets M, N, we denote by 2^M the powerset of M, by $M \uplus N$ their disjoint union and by $[M \to N]$ the set of functions from M to N.

Given a relation $\Re \subseteq M \times M$, the $\Re$-closure of a set $N \subseteq M$ is the set $N^{\Re} = \{m \in M \mid \exists n \in N, (n, m) \in \Re\}$; we say that N is $\Re$-closed iff $N^{\Re} \subseteq N$. If $\Sigma \subseteq 2^M$, then $\Sigma(\Re)$ denotes the set of $\Re$-closed elements of Σ.

A set $\Sigma \subseteq 2^M$ is a *σ-algebra over M* if it contains M and it is closed under complement and countable union.

Given a σ-algebra Σ over M, the tuple (M, Σ) is called *measurable space* and the elements of Σ, *measurable sets*. A set $\Omega \subseteq 2^M$ is a *generator for* Σ if Σ is the closure of Ω under complement and countable union.

Given a measurable space (M, Σ), a function $\mu : \Sigma \to \mathbb{R}^+$ is a measure iff $\mu(\emptyset) = 0$ and for any sequence $\{N_i \mid i \in I \subseteq \mathbb{N}\}$ of pairwise disjoint measurable sets, $\mu(\bigcup_{i \in I} N_i) = \sum_{i \in I} \mu(N_i)$. The set of all measures on (M, Σ) is denoted by $\Delta(M, \Sigma)$.

We organize $\Delta(M, \Sigma)$ as a measurable space by considering the σ-algebra $\mathfrak{F}$ generated, for arbitrary $S \in \Sigma$ and $r > 0$, by the sets $F_S^r = \{\mu \in \Delta(M, \Sigma) : \mu(S) \geq r\}$.

Given two measurable spaces (M, Σ) and (N, Σ'), a mapping $f : M \to N$ is *measurable* if for any $T \in \Sigma', f^{-1}(T) \in \Sigma$. We use $[\![M \to N]\!]$ to denote the class of measurable mappings from (M, Σ) to (N, Σ'), assuming of course that Σ and Σ' are clear from the context.

Central for this paper is the notion of *analytic space* that supports some of the main results. As properties of analytic spaces are however not used here directly, we only recall the main definitions. For detailed discussion on this topic related to Markov processes, the reader is referred to [Pan09] (Section 7.5) or to [Dob07] (Section 4.4).

A metric space (M, d) is *complete* if every Cauchy sequence converges in M. A *Polish space* is the topological space underlying a complete metric space with a countable dense subset. An *analytic space* is the image of a Polish space under a continuous function between Polish spaces.

III. Continuous Markov processes

In this section we introduce continuous Markov processes (CMPs) [CLM11a], [CLM11b] which are models of stochastic systems with analytic state space and continuous-time transitions. The definition is similar to the one proposed by Desharnais and Panangaden in [DP03], but it exploits an equivalence between the definitions of Harsanyi type spaces [MV04] and a coalgebraic view of labelled Markov processes [dVR99] proved, for instance, by Doberkat in [Dob07]. However, with respect to [CLM11a], [CLM11b] or to [Pan09], [DEP02], [Dob07], we do not consider action labels. The labels can easily be added without changing any aspects of the theory.

Definition 1 (Continuous Markov processes)**.** *Given an analytic set (M, Σ), where Σ is the Borel σ-algebra generated by the topology, a* continuous Markov kernel *(CMK) is a tuple $\mathcal{M} = (M, \Sigma, \theta)$, where $\theta \in [\![M \to \Delta(M, \Sigma)]\!]$ is the* transition function. *The set M is the* support set of $\mathcal{M}$ *denoted $supp(\mathcal{M})$.*

If $m \in M$, then $(\mathcal{M}, m)$ is a continuous Markov process.

Notice that θ is a measurable mapping between (M, Σ) and $(\Delta(M, \Sigma), \mathfrak{F})$, where $\mathfrak{F}$ is the sigma algebra on $\Delta(M, \Sigma)$ defined in the preliminaries. This condition on θ is equivalent with the conditions on the two-variable *rate function* used in [Pan09], [DEP02], [DP03] to define continuous Markov processes.

A. Bisimulation

Stochastic bisimulation for CMPs follows the line of Larsen-Skou probabilistic bisimulation [LS91], [DEP02], [Pan09]. Recall, from the preliminary section, that $\Sigma(\Re)$ in the next definition denotes the $\Re$-closed sets of Σ.

Definition 2 (Stochastic Bisimulation)**.** *Given a CMK $\mathcal{M} = (M, \Sigma, \theta)$ a binary relation $\Re \subseteq M \times M$ is a* stochastic bisimilarity relation *if whenever $(m, n) \in \Re$, for any $C \in \Sigma(\Re)$,*

$$\theta(m)(C) = \theta(n)(C).$$

Two processes $(\mathcal{M}, m)$ and $(\mathcal{M}, n)$ are stochastic bisimilar*, written $m \sim_{\mathcal{M}} n$, if they are related by a stochastic bisimilarity relation.*

Observe that, for any CMK $\mathcal{M}$ there exist stochastic bisimulation relations: for instance, the identity relation on its support-set is such a relation. The relation $\sim_{\mathcal{M}}$ is the largest stochastic bisimulation relation.

Definition 3 (Disjoint union)**.** *If $\mathcal{M} = (M, \Sigma, \theta)$ and $\mathcal{M}' = (M', \Sigma', \theta')$ are CMKs, then $\mathcal{M}'' = (M'', \Sigma'', \theta'')$ defined by $M'' = M \uplus M'$, Σ'' is the σ-algebra generated by $\Sigma \uplus \Sigma'$ and*

$$\theta''(m)(N \uplus N') = \begin{cases} \theta(m)(N) & \text{if } m \in M \\ \theta'(m)(N') & \text{if } m \in M' \end{cases}$$

for arbitrary $N \in \Sigma$ and $N' \in \Sigma'$, is the disjoint union of $\mathcal{M}$ and $\mathcal{M}'$ denoted by $\mathcal{M}'' = \mathcal{M} \uplus \mathcal{M}'$.

Observe that the disjoint union of CMKs is a CMK.

The previous definition allows us to define stochastic bisimulation between processes from different CMKs. If $m \in M$ and $m' \in M'$, we say that $(\mathcal{M}, m)$ and $(\mathcal{M}', m')$ are *bisimilar* written $(\mathcal{M}, m) \sim (\mathcal{M}', m')$ whenever $m \sim_{\mathcal{M} \uplus \mathcal{M}'} m'$.

B. Generators

The definition of bisimulation can be amended to focus on two particular classes of generators of the σ-algebra.

Definition 4 (Bisimulation Generators)**.** *Consider the CMK $\mathcal{M} = (M, \Sigma, \theta)$ and let*

$$\Theta = \{\theta(m)^{-1}([0, r]) \mid m \in M, r \in \mathbb{R}^+\}.$$

- *The* bisimulation generator of $\mathcal{M}$, *denoted by* $G_{\mathcal{M}}$, *is the closure of* Θ *under union and intersection.*
- *The* extended bisimulation generator of $\mathcal{M}$, *denoted* $\overline{G_{\mathcal{M}}}$, *is the closure of* Θ *under union, intersection and complement.*

Observe that the bisimulation generators are generators of $\Sigma(\sim)$ which is a sub-sigma algebra of Σ, i.e., $\Sigma(\sim)$ is the closure of both $G_{\mathcal{M}}$ and $\overline{G_{\mathcal{M}}}$ under complement and countable union. This observation allows us to characterize the stochastic bisimulation from the perspective of the bisimulation generators and to propose some generalizations of the stochastic bisimulation. But more importantly, they will be instrumental later, for obtaining our results on logical characterization.

Theorem 1. *Given a CMK* $\mathcal{M} = (M, \Sigma, \theta)$*, a relation* $\mathfrak{R} \subseteq M \times M$ *is a stochastic bisimilarity relation iff one (or both) of the following equivalent conditions is satisfied.*

- *whenever* $(m, n) \in \mathfrak{R}$, $\theta(m)(C) = \theta(n)(C)$ *for any* $C \in G_{\mathcal{M}}$,
- *whenever* $(m, n) \in \mathfrak{R}$, $\theta(m)(C) = \theta(n)(C)$ *for any* $C \in \overline{G_{\mathcal{M}}}$.

Proof: It is sufficient to notice that, because $G_{\mathcal{M}}$ and $\overline{G_{\mathcal{M}}}$ are generators for $\Sigma(\sim)$, for any $C \in G_{\mathcal{M}}$ (any $C \in \overline{G_{\mathcal{M}}}$), $\theta(m)(C) = \theta(n)(C)$ iff for any $C \in \Sigma(\sim)$, $\theta(m)(C) = \theta(n)(C)$. ■

IV. Continuous Markovian Logics

In this section we recall the continuous Markovian logic (CML) introduced and studied in [CLM11a], [CLM11b]. This logic extends the probabilistic logics for discrete-time Markov processes [LS91], [DEP02], [Pan09] and for Harsanyi type spaces [FH94], [Zho07] to stochastic domains and it characterizes the stochastic bisimulation. In addition to the Boolean operators, this logic is endowed with *stochastic modal operators* that approximate the rates of transitions. In the original definition, $L_r\phi$ is a property of a CMP $(\mathcal{M}, m)$ whenever the rate of the transition from m to the class of the states satisfying ϕ is *at least* r.

A. Syntax

Definition 5 (Syntax)**.** *The set* $\mathcal{L}$ *of formulas of CML is generated by the following grammar, for arbitrary* $r \in \mathbb{Q}^+$.

$$\mathcal{L}: \quad \phi := \top \mid \neg\phi \mid \phi \wedge \phi \mid L_r\phi.$$

As usual, we work with all the Boolean operators, including $\bot = \neg\top$. In addition, we isolate two useful sublanguages of $\mathcal{L}$:

$$\mathcal{L}^+: \quad \psi := \top \mid \psi \wedge \psi \mid \psi \vee \psi \mid L_r\psi,$$

$$\mathcal{L}^- = \{\neg\phi \mid \phi \in \mathcal{L}^+\}.$$

B. Parametrized Semantics: ε-satisfiability

In[CLM11a], [CLM11b] the first two authors in collaboration with Cardelli define the semantics of CML for arbitrary CMPs, henceforth the *classic semantics* for CML. We will take a similar approach in this paper with the only difference that the satisfiability relation is parametrized. Thus, for each rational $\varepsilon \geq 0$, we introduce an ε-semantics that provides an approximation of the classic semantics. The ε-semantics can be seen as an "approximation from below" of the classic semantics: while $L_r\phi$ is interpreted at m as *"the rate of the transitions from m to the class of the states satisfying ϕ is at least r"*, in the ε-semantics it means that *"the rate of the transitions from m to the class of the states satisfying ϕ is at least $r - \varepsilon$"*. In this way one can encode observational errors in the logic. Unlike the similar approach of [DLT08], we propose a Boolean semantics.

Definition 6 (ε-Satisfiability)**.** *For an arbitrary rational* $\varepsilon \geq 0$*, the* ε-satisfiability *relation* $\models_\varepsilon \subseteq \mathfrak{M} \times \mathcal{L}$ *is defined inductively on the structure of* $\phi \in \mathcal{L}$*, as follows.*

- $m \models_\varepsilon \top$ *always,*
- $m \models_\varepsilon \neg\phi$ *iff it is not the case that* $m \models_\varepsilon \phi$,
- $m \models_\varepsilon \phi \wedge \psi$ *iff* $m \models_\varepsilon \phi$ *and* $m \models_e \psi$,
- $m \models_\varepsilon L_r\phi$ *iff* $\theta(m)([\![\phi]\!]_\varepsilon) + \varepsilon \geq r$,

where $[\![\phi]\!]_\varepsilon = \{m \in \mathfrak{M} \mid m \models_\varepsilon \phi\}$.

Notice that the classic semantics for CML introduced in [CLM11a], [CLM11b] is nothing else but 0-semantics, since $\models_0 = \models$.

Example 1. *Consider the CMP* $\mathcal{M} = (M, 2^M, \theta)$ *represented in Figure 1, where* $M = \{m, m_1, m_2, m_3, m_4, m_5\}$ *and* θ *is defined by the values* $r, s, s', u \in \mathbb{Q}^+$ *that label the transition arrows*[1]*. We can now understand the difference between the classic and the* ε*-semantics. For instance,*

$$m \models L_{s+s'}L_u\top$$

since $m_2 \sim m_4$, $\theta(m)(\{m_2, m_4\}) = s + s'$ *and* $m_2 \models L_u\top$ *because* $m_3 \sim m_5$ *and* $\theta(m_2)(\{m_3, m_5\}) = u$.

Similarly, for some $\varepsilon > 0$,

$$m \models_\varepsilon L_{s+s'+\varepsilon}L_{u+\varepsilon}\top$$

since $\theta(m)(\{m_2, m_4\}) = s + s' \geq (s + s' + \varepsilon) - \varepsilon$ *and* $m_2 \models_\varepsilon L_{u+\varepsilon}\top$ *because* $\theta(m_2)(\{m_3, m_5\}) = u \geq (u + \varepsilon) - \varepsilon$.

On the other hand,

$$m \not\models L_{s+s'+\varepsilon}L_{u+\varepsilon}\top$$

since $\theta(m)(\{m_2, m_4\}) = s + s' \not\geq (s + s' + \varepsilon)$ *and* $m_2 \not\models L_{u+\varepsilon}\top$ *because* $\theta(m_2)(\{m_3, m_5\}) = u \not\geq (u + \varepsilon)$.

And this is exactly the way in which we see ε *as an observational error.*

The semantics of $L_r\phi$ is well defined only if $[\![\phi]\!]_\varepsilon$ is measurable. This is guaranteed by the fact that θ is a measurable mapping between (M, Σ) and $(\Delta(M, \Sigma), \mathfrak{F})$, as proved in the next lemma.

Lemma 1. *For any* $\phi \in \mathcal{L}$, $[\![\phi]\!]_\varepsilon \in \Sigma$.

Proof: Induction on ϕ. The Boolean cases derive from the fact that $[\![\top]\!] = supp(\mathcal{M})$ is measurable and that the set of measurable sets is closed to complement and finite intersection.

[1] For simplicity we only represented the transitions with strict positive rates.

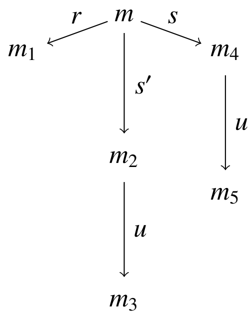

Fig. 1. A Markov process

Consider now the case $\phi = L_r\psi$. The inductive hypothesis guarantees that $[\![\psi]\!]_\varepsilon \in \Sigma$, hence, $\{\mu \in \Delta(M,\Sigma) | \mu([\![\psi]\!]_\varepsilon) \geq r - \varepsilon\}$ is measurable in $\Delta(M,\Sigma)$. Because θ is a measurable mapping, we obtain that $[\![L_r\psi]\!]_\varepsilon = (\theta)^{-1}(\{\mu \in \Delta(M,\Sigma) | \mu([\![\psi]\!]_\varepsilon) \geq r - \varepsilon\})$ is measurable. ■

We extend the classical metalogical concepts to the parametric metatheory.

Definition 7. *Given a rational* $\varepsilon \geq 0$*, a formula* ϕ *is* ε-satisfiable *if there exists* $m \in supp(\mathcal{M})$ *such that* $m \models_\varepsilon \phi$*. We say that* ϕ *is* ε-valid*, denoted by* $\models_\varepsilon \phi$*, if* $\neg\phi$ *is not* ε*-satisfiable.*

For notational convenience we will write $m \not\models_\varepsilon \phi$ when it is not the case that $m \models_\varepsilon \phi$, and use $\models$ in place of $\models_0$.

The proof of the previous lemma reveals a deeper result connecting the ε-semantics, if we involve our notion of bisimulation generators.

Corollary 1. *For any rational* $\varepsilon \geq 0$*,* $G_\mathcal{M} = \{[\![\phi]\!]_\varepsilon \mid \phi \in \mathcal{L}^+\}$ *and* $\overline{G_\mathcal{M}} = \{[\![\phi]\!]_\varepsilon \mid \phi \in \mathcal{L}\}$*.*

Proof: It is sufficient to notice that the elements of $\mathcal{L}^+$ are formed from $\top$ and formulas of type $L_r\phi$ by taking finite disjunctions and conjunctions, while the elements of $\mathcal{L}$ are formed from $\top$ and formulas of type $L_r\phi$ by taking finite disjunctions, finite conjunctions and negation. Since $[\![\phi \wedge \psi]\!]_\varepsilon = [\![\phi]\!]_\varepsilon \cap [\![\psi]\!]_\varepsilon$, $[\![\phi \vee \psi]\!]_\varepsilon = [\![\phi]\!]_\varepsilon \cup [\![\psi]\!]_\varepsilon$ and $[\![\neg\phi]\!]_\varepsilon = supp(\mathcal{M}) \setminus [\![\phi]\!]_\varepsilon$, the results derive directly from the definitions of $G_\mathcal{M}$ and $\overline{G_\mathcal{M}}$ and the fact that

$$[\![L_r\psi]\!]_\varepsilon = (\theta)^{-1}(\{\mu \in \Delta(M,\Sigma) | \mu([\![\psi]\!]_\varepsilon) \geq r - \varepsilon\}).$$

■

The major advantage that the parametric semantics provides is that one can handle in parallel properties from different semantics and, for instance, can prove $\varepsilon + \varepsilon'$-satisfiability properties from properties concerning ε-satisfiability.

In what follows we establish a few such results. The first lemma establishes the relation between ε-semantics and the classic semantics.

Lemma 2. *If* $\phi \in \mathcal{L}^+$*, then for arbitrary* $\varepsilon, \varepsilon' \geq 0$*,* $m \models_\varepsilon \phi$ *implies* $m \models_{\varepsilon+\varepsilon'} \phi$*. In particular,* $m \models \phi$ *implies* $m \models_\varepsilon \phi$*.*

Proof: Induction on $\phi \in \mathcal{L}^+$. The only nontrivial case, since we work with positive formulas, is $\phi = L_r\psi$. From $m \models_\varepsilon L_r\psi$ we obtain that $\theta(m)([\![\psi]\!]_\varepsilon) \geq r - \varepsilon$. From the inductive hypothesis we have that $[\![\psi]\!]_\varepsilon \subseteq [\![\psi]\!]_{\varepsilon+\varepsilon'}$. Hence, $\theta(m)([\![\psi]\!]_{\varepsilon+\varepsilon'}) \geq r - (\varepsilon + \varepsilon')$. ■

The following counter-example shows that we cannot hope for the result to hold for negative formulas.

Example 2. *Consider the CMK* $\mathcal{M} = (\{m\}, 2^{\{m\}}, \theta)$ *with* $\theta(m)([\![\top]\!]) = r$*. Clearly* $m \models \neg L_{r+\delta}\top$ *for all* $\delta > 0$*. Suppose that we also have* $m \models_\varepsilon \neg L_{r+\delta}\top$*. This is equivalent to* $m \models \neg L_{r+\delta-\varepsilon}\top$*, i.e.,* $\theta(m)([\![\top]\!]) < r + \delta - \varepsilon$ *for all* $\delta > 0$*. This last inequality implies* $\theta(m)([\![\top]\!]) \leq r - \varepsilon$ *which contradict our initial assumption.*

Notice that if ε is growing, the set $[\![\phi]\!]_\varepsilon$ is increasing when $\phi \in \mathcal{L}^+$ and is decreasing when $\phi \in \mathcal{L}^-$.

Although negation turns out to be problematic in the case of the previous lemma, we can however characterize the relation between $\models_\varepsilon$ and $\models_{\varepsilon+\varepsilon'}$ for the entire language. To characterize completely the relation between two parametric semantics, we define a pair of dual encodings.

Definition 8. *Let* $\langle\ \rangle_\varepsilon : \mathcal{L} \to \mathcal{L}$ *and* $\langle\ \rangle^\varepsilon : \mathcal{L} \to \mathcal{L}$ *be two functions on* $\mathcal{L}$ *defined as follows.*

$\langle\top\rangle_\varepsilon = \top$	$\langle\top\rangle^\varepsilon = \top$
$\langle\phi \wedge \psi\rangle_\varepsilon = \langle\phi\rangle_\varepsilon \wedge \langle\psi\rangle_\varepsilon$	$\langle\phi \wedge \psi\rangle^\varepsilon = \langle\phi\rangle^\varepsilon \wedge \langle\psi\rangle^\varepsilon$
$\langle\neg\phi\rangle_\varepsilon = \neg\langle\phi\rangle_\varepsilon$	$\langle\neg\phi\rangle^\varepsilon = \neg\langle\phi\rangle^\varepsilon$
$\langle L_r\phi\rangle_\varepsilon = L_{r \dot{-} \varepsilon}\langle\phi\rangle_\varepsilon$	$\langle L_r\phi\rangle^\varepsilon = L_{r+\varepsilon}\langle\phi\rangle^\varepsilon$

where $r \dot{-} \varepsilon = \max\{0, r - \varepsilon\}$

Observe that for any ϕ that is not of type $L_r\psi$ with $r < \varepsilon$, we have that $\langle\langle\phi\rangle^\varepsilon\rangle_\varepsilon = \langle\langle\phi\rangle_\varepsilon\rangle^\varepsilon = \phi$.

Before we turn to the main theorem of this section, we apply the previous definition to obtain a result on limits. The result may additionally be considered a form of reverse implication for Lemma 2.

Lemma 3. *If* $\phi \in \mathcal{L}^+$ *and for every rational* $\varepsilon > 0$*,* $m \models_{\varepsilon'+\varepsilon} \phi$*, then* $m \models_{\varepsilon'} \phi$*. In particular, if* $m \models_\varepsilon \phi$ *for all rationals* $\varepsilon > 0$*, then also* $m \models \phi$*.*

Proof: We prove it by induction on $\phi \in \mathcal{L}^+$. The cases $\phi = \top$, $\phi = \phi_1 \wedge \phi_2$ and $\phi = \phi_1 \vee \phi_2$ are trivial.
The case $\phi = L_r\psi$: $m \models_{\varepsilon'+\varepsilon} L_r\psi$ implies $m \models_{\varepsilon'} L_{r-\varepsilon}\langle\psi\rangle_\varepsilon$ which is equivalent to $\theta(m)([\![\langle\psi\rangle_\varepsilon]\!]_{\varepsilon'}) \geq (r - \varepsilon) - \varepsilon'$ and this is so for any $\varepsilon > 0$. Then $\lim_{\varepsilon\to 0} \theta(m)([\![\langle\psi\rangle_\varepsilon]\!]_{\varepsilon'}) \geq \lim_{\varepsilon\to 0}((r-\varepsilon)-\varepsilon')$ implying $\theta(m)([\![\psi]\!]_{\varepsilon'}) \geq r - \varepsilon'$. Consequently, $m \models_{\varepsilon'} L_r\psi$. ■

We are now ready to state the main theorem of this section that establishes the relation between various parametrized semantics for the entire language $\mathcal{L}$.

Theorem 2. *For arbitrary* $\phi \in \mathcal{L}$*,*

1) $m \models_{\varepsilon+\varepsilon'} \phi$ *iff* $m \models_\varepsilon \langle\phi\rangle_{\varepsilon'}$*,*
2) $m \models_\varepsilon \phi$ *iff* $m \models_{\varepsilon+\varepsilon'} \langle\phi\rangle^{\varepsilon'}$*.*

Proof: 1. Induction on ϕ.
As before, the Boolean cases are trivial.
The case $\phi = L_r\psi$: $m \models_{\varepsilon+\varepsilon'} L_r\psi$ iff $\theta(m)([\![\psi]\!]_{\varepsilon+\varepsilon'}) \geq r-(\varepsilon+\varepsilon')$. From the inductive hypothesis we have that $[\![\psi]\!]_{\varepsilon+\varepsilon'} = [\![\langle\psi\rangle_{\varepsilon'}]\!]_{\varepsilon}$. Hence, $m \models_{\varepsilon+\varepsilon'} L_r\psi$ iff $\theta(m)([\![\langle\psi\rangle_{\varepsilon'}]\!]_{\varepsilon}) \geq (r-\varepsilon')-\varepsilon$ which is further equivalent to $m \models_{\varepsilon} L_{r-\varepsilon'}\langle\psi\rangle_{\varepsilon'}$.

2. Induction on ϕ.
The Boolean cases are trivial.
The case $\phi = L_r\psi$: $m \models_{\varepsilon} L_r\psi$ iff $\theta(m)([\![\psi]\!]_{\varepsilon}) \geq r-\varepsilon$. From the inductive hypothesis we have that $[\![\psi]\!]_{\varepsilon} = [\![\langle\psi\rangle^{\varepsilon'}]\!]_{\varepsilon+\varepsilon'}$. Hence, $m \models_{\varepsilon} L_r\psi$ iff $\theta(m)([\![\langle\psi\rangle^{\varepsilon'}]\!]_{\varepsilon+\varepsilon'}) \geq r-\varepsilon = (r+\varepsilon')-(\varepsilon+\varepsilon')$ which is further equivalent to $m \models_{\varepsilon+\varepsilon'} L_{r+\varepsilon'}\langle\psi\rangle^{\varepsilon'}$. ∎

From this last theorem we can derive a characterization of the relation between the classic semantics and the ε-semantics.

Corollary 2. *For arbitrary $\phi \in \mathcal{L}$,*

1) $m \models_{\varepsilon} \phi$ *iff* $m \models \langle\phi\rangle_{\varepsilon}$,
2) $m \models \phi$ *iff* $m \models_{\varepsilon} \langle\phi\rangle^{\varepsilon}$.

V. Parametrized Proof Theory: ε-Provability

In this section we extend the metatheory and define a *parametrized proof system* for our logic that corresponds to the parametrized semantics. The parametrized proof system will permit us to prove, syntactically, approximated properties of models. We should emphasize that we will not work with "approximated proofs", but with "exact proofs" about "approximated properties" and this is where the Boolean character of our metatheory plays its role.

For each rational $\varepsilon \geq 0$ we introduce a notion of ε-provability denoted by $\vdash_{\varepsilon}$. Table I contains a Hilbert-style axiomatization of ε-provability for our ε-semantics. The axioms and rules, which are considered in addition to the axiomatization of classic propositional logic, are stated for propositional variables $\phi, \psi \in \mathcal{L}$ and arbitrary $s, r \in \mathbb{Q}^+$.

(A1):	$\vdash_{\varepsilon} L_{\varepsilon}\phi$
(A2):	$\vdash_{\varepsilon} L_{r+s}\phi \to L_r\phi$
(A3):	$\vdash_{\varepsilon} L_r(\phi\wedge\psi) \wedge L_s(\phi\wedge\neg\psi) \to L_{r+s-\varepsilon}\phi$
(A4):	$\vdash_{\varepsilon} \neg L_r(\phi\wedge\psi) \wedge \neg L_s(\phi\wedge\neg\psi) \to \neg L_{r+s-\varepsilon}\phi$
(R1):	If $\vdash_{\varepsilon} \phi \to \psi$ then $\vdash_{\varepsilon} L_r\phi \to L_r\psi$
(R2):	$\{L_r\phi \mid \text{r} < s\} \vdash_{\varepsilon} L_s\phi$
(R3):	$\{L_r\phi \mid \text{r} > s\} \vdash_{\varepsilon} \bot$

TABLE I
The axiomatization of ε-provability for CML

Axiom (A1) guarantees that the rate of any transition with an ε-approximation is at least $0+\varepsilon=\varepsilon$; this encodes the fact that the real measure of any set cannot be negative. (A2) states that if a rate is at least $r+s$ then it is at least r. (A3) and (A4) encode the additive properties of measures for disjoint sets: $[\![\phi\wedge\psi]\!]_{\varepsilon}$ and $[\![\phi\wedge\neg\psi]\!]_{\varepsilon}$ are disjoint sets of processes such that $[\![\phi\wedge\psi]\!]_{\varepsilon} \cup [\![\phi\wedge\neg\psi]\!]_{\varepsilon} = [\![\phi]\!]_{\varepsilon}$. The rule (R1) establishes the monotonicity of L_r^a. In this axiomatic system we have two infinitary rules, (R2) and (R3). The first reflects the Archimedian property of rationals: if it is possible a transition from a state to a given set of states at any rate $r < s$, then the rate of the transition is at least s. (R3) eliminates the possibility of having transitions at infinite rates.

Now we can complete the list of parametric meta-concepts initiated in Definition 7.

Definition 9. *A formula $\phi \in \mathcal{L}$ is* ε-provable*, written $\vdash_{\varepsilon} \phi$, if either it is an instance of an axiom or it can be proved from axioms using the proof rules. A formula $\phi \in \mathcal{L}$ is* ε-consistent*, if $\phi \to \bot$ is not provable.*

Given a set $\Phi \subseteq \mathcal{L}$ of formulas, we say that Φ ε-proves ϕ, denoted by $\Phi \vdash_{\varepsilon} \phi$, if ϕ can be proved from axioms and the formulas of Φ. Φ is ε-consistent *if it is not the case that $\Phi \vdash_{\varepsilon} \bot$.*

*For a sublanguage $\mathcal{L}' \subseteq \mathcal{L}$, we say that $\Phi \in \mathcal{L}$ is $\mathcal{L}'$-*maximally ε-consistent *if Φ is ε-consistent and no formula of $\mathcal{L}'$ can be added to it without making it ε-inconsistent.*

The next theorem states that $\models_{\varepsilon}$ and $\vdash_{\varepsilon}$ agree about the class of CMPs. As before, we will simply denote $\vdash_0$ by $\vdash$ and we call to it as *classic provability*.

Theorem 3 (Soundness and Weak Completeness)**.** *The axiomatic system of ε-provability is sound and complete for the ε-semantics, i.e., for any $\phi \in \mathcal{L}$,*

$$\vdash_{\varepsilon} \phi \text{ iff } \models_{\varepsilon} \phi.$$

Proof: In [CLM11a] we have shown that in table II we have a sound and complete axiomatizaion of the classic provability for the classic semantics. In other words, we have proved that $\vdash \phi$ iff $\models \phi$.

(B1):	$\vdash L_0\phi$
(B2):	$\vdash L_{r+s}\phi \to L_r\phi$
(B3):	$\vdash L_r(\phi\wedge\psi) \wedge L_s(\phi\wedge\neg\psi) \to L_{r+s}\phi$
(B4):	$\vdash \neg L_r(\phi\wedge\psi) \wedge \neg L_s(\phi\wedge\neg\psi) \to \neg L_{r+s}\phi$
(S1):	If $\vdash \phi \to \psi$ then $\vdash L_r\phi \to L_r\psi$
(S2):	$\{L_r\phi \mid r < s\} \vdash L_s\phi$
(S3):	$\{L_r\phi \mid r > s\} \vdash \bot$

TABLE II
The axiomatic system of classic provability

Obviously, the axioms (A1)-(A4) are the $\langle\ \rangle_{\varepsilon}$-encodings of the axioms (B1)-(B4) and similarly (R1)-(R3) are the encodings of (S1)-(S3).

Consequently, we obtain that $\vdash_{\varepsilon} \phi$ iff $\vdash \langle\phi\rangle_{\varepsilon}$ and $\vdash \phi$ iff $\vdash_{\varepsilon} \langle\phi\rangle^{\varepsilon}$. Now, in the light of Theorem 2 we obtain $\vdash_{\varepsilon} \phi$ iff $\vdash \langle\phi\rangle_{\varepsilon}$ iff $\models \langle\phi\rangle_{\varepsilon}$ iff $\models_{\varepsilon} \phi$. ∎

Trivial consequences of the completeness theorem, Theorem 2 and Lemma 2 are comprised in the next lemma which establishes the relation between various ε-provabilities.

Lemma 4. *For arbitrary $\phi \in \mathcal{L}$, and rationals $\varepsilon, \varepsilon' \geq 0$,*

1) $\vdash_{\varepsilon+\varepsilon'} \phi$ *iff* $\vdash_{\varepsilon} \langle\phi\rangle_{\varepsilon'}$*; in particular,* $\vdash_{\varepsilon} \phi$ *iff* $\vdash \langle\phi\rangle_{\varepsilon}$.

2) $\vdash_\varepsilon \phi$ *iff* $\vdash_{\varepsilon+\varepsilon'} \langle\phi\rangle^{\varepsilon'}$*; in particular,* $\vdash \phi$ *iff* $\vdash_\varepsilon \langle\phi\rangle^\varepsilon$.
3) *If* $\phi \in \mathcal{L}^+$, $\vdash_{\varepsilon'} \phi$ *implies* $\vdash_{\varepsilon+\varepsilon'} \phi$.

The previous lemma allows us to prove a parameterized deduction theorem that establishes the frame in which one can use various ε-provabilities relations in the same proof.

Theorem 4 (Parameterized Deduction Theorem)**.** *For positive rationals* ε *and* ε'*, and* $\phi \in \mathcal{L}^+$*,*

1) *if* $\vdash_{\varepsilon'+\varepsilon} (\phi \to \psi)$ *and* $\vdash_{\varepsilon'} \phi$*, then* $\vdash_{\varepsilon'+\varepsilon} \psi$*;*
2) *if* $\vdash_{\varepsilon'+\varepsilon} (\neg\psi \to \neg\phi)$ *and* $\vdash_{\varepsilon'} \phi$*, then* $\vdash_{\varepsilon'+\varepsilon} \psi$.

Proof: The result is a consequence of Modus Ponens in the light of the result of case 3 of Lemma 4. ■

The relation between the classic semantics and the ε-semantics also allows us to prove the next decidability result.

Theorem 5 (Decidability and Complexity of ε-satisfiability)**.** *The problem of deciding if an arbitrary property* $\phi \in \mathcal{L}$ *is* ε*-satisfiable, i.e., if there exists a CMP* m *such that* $m \models_\varepsilon \phi$*, is decidable in* PSPACE.

Proof: It is sufficient to observe that if there exists m such that $m \models_\varepsilon \phi$, then $m \models \langle\phi\rangle_\varepsilon$. Hence the ε-satisfiability problem for ϕ can be reduced to the classic satisfiability problem for $\langle\phi\rangle_\varepsilon$ which has been proved decidable in PSPACE in [KP10]. The complexity class is the same since the encoding $\langle\ \rangle_\varepsilon$ is linear. ■

Following the same proof line of Theorems 3 and 4 we can prove that the logic enjoys a parametrized version of finite model property.

Theorem 6 (Finite model property)**.** *Given an arbitrary* ε*-consistent formula* $\phi \in \mathcal{L}$*, there exists a finite CMP* $(\mathcal{M}, m)$ *such that* $m \models_\varepsilon \phi$.

Proof: In [CLM11a] we have proved the finite model property for $\varepsilon = 0$. More exactly we have proved that if $\phi \in \mathcal{L}$ is consistent, then there exists a CMP $(\mathcal{M}, m)$ such that $\mathcal{M}, m \models \phi$.

If $\phi \in \mathcal{L}$ is ε-consistent, using Lemma 4 we obtain that $\langle\phi\rangle_\varepsilon$ is consistent. Using the finite model property for the classic case we get that there exists a CMP $(\mathcal{M}, m)$ such that $m \models \langle\phi\rangle_\varepsilon$. Now we apply again Corollary 2 and obtain $m \models_\varepsilon \phi$. ■

VI. Behavioral Properties

In the previous sections we developed the parametric metatheory and prove that it enjoys most of the metaproperties of the classic metatheory. In this section we investigate the relationship between this parametric logical framework and the behavioral properties of CMPs. We begin by recalling a result proved in [CLM11a].

Theorem 7 (Logical characterization of bisimulation)**.** *Let* $\mathcal{M} = (M, \Sigma, \tau)$ *be a CMK and* $m, m' \in M$*. The following assertions are equivalent.*

1) $m \sim m'$*;*
2) *For any* $\phi \in \mathcal{L}$*,* $m \models \phi$ *iff* $m' \models \phi$*;*
3) *For any* $\phi \in \mathcal{L}^+$*,* $m \models \phi$ *iff* $m' \models \phi$.

Because the encodings $\langle\ \rangle_\varepsilon$ and $\langle\ \rangle^\varepsilon$ preserve the logical implication, a consequence of the fact that CML characterizes stochastic bisimulation is the next theorem.

Theorem 8 (Parametrized characterization of bisimulation)**.** *For arbitrary* $\varepsilon \geq 0$*, the following assertions are equivalent.*

1) $m \sim m'$*;*
2) *For any* $\phi \in \mathcal{L}$*,* $m \models_\varepsilon \phi$ *iff* $m' \models_\varepsilon \phi$*;*
3) *For any* $\phi \in \mathcal{L}^+$*,* $m \models_\varepsilon \phi$ *iff* $m' \models_\varepsilon \phi$.

However, the interrelations between various ε-semantics allow us to prove some stronger results. For this, in what follows, we extend the concept of bisimulation towards a notion of ε-orders that reflect the approximated behaviors. Similar notions have been proposed in literature for less general models. It is the case of the point-wise simulation distance defined in [**?**] for discrete probabilistic systems and similarly in [AHM03], [AFS09] for weighted systems, and for general probabilistic systems in [vBW01], [D+04].

Recall that for a set C and a relation R, C^R denotes the closure of C to R (see the section Preliminary definitions).

Definition 10 (ε-behavioral orders)**.** *Given an CMK* $\mathcal{M} = (M, \Sigma, \theta)$*, a relation* $R \subseteq M \times M$ *closed under bisimulation is*

- *an* ε-behavioral order *whenever* $m\ R\ n$*, implies that for any* $C \in G_\mathcal{M}$*,*
$$\theta(n)(C) - \theta(m)(C^R) \leq \varepsilon.$$
- *an* essential ε-behavioral order *whenever* $m\ R\ n$*, implies that for any* $C \in \overline{G_\mathcal{M}}$*,*
$$\theta(n)(C) - \theta(m)(C^R) \in [0, \varepsilon].$$

We use $\prec_e$ to denote the largest ε-behavioral order and $\prec^+_\varepsilon$ to denote the largest essential ε-behavioral order.

Observe that, in the definition of ε-behavioural order we can have $\theta(n)(C) < \theta(m)(C^R)$, while for essential ε-behavioural order we have always that $\theta(n)(C) \geq \theta(m)(C^R)$. Notice also that both $\prec_\varepsilon$ and $\prec^+_\varepsilon$ are not equivalences and that an essential ε-behavioral order is an ε-behavioral order, i.e., $\prec^+_\varepsilon \subseteq \prec_\varepsilon$.

Example 3. *Figure 2 shows three discrete processes with initial states m,n and o respectively. For simplicity, we have not represented the transitions with rate* 0*. Theorem 11 shows that* $\prec_\varepsilon$ *induces a pseudometric on CMPs.*

Assuming that $s + s' = t$*, we obtain*

$$R = \{(m, n), (m_1, n_1), (m_2, n_2), (m_4, n_2), (m_3, n_3), (m_5, n_3)\} \subseteq \prec_{2\varepsilon},$$

i.e., $m \prec_{2\varepsilon} n$ *and the value* 2ε *is obtained from*

$$\theta(n)(C) - \theta(m)(C^R) = 2\varepsilon$$

for $C = \{m_1, m_2, m_4, n_2, n_2\}$*. Observe in this case that* $m_4 \prec_{2\varepsilon} n_2$ *even if the rate of exiting* n_2 *in smaller than the rate of exiting* m_4*. But for this reason we do not have* $m \prec^+_{\varepsilon'} n$ *for all* $e' > 0$.

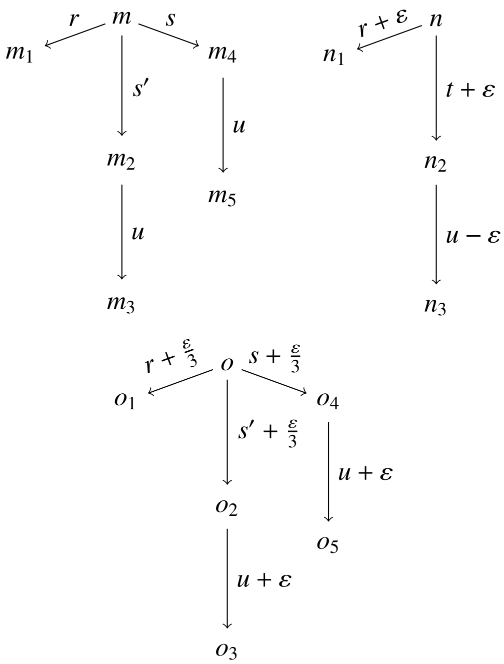

Fig. 2. Systems demonstrating ε-behavioral orders and the significance of stochastic transitions.

Similarly, we obtain $m \prec_\varepsilon^+ o$ for

$$R = \{(m, o), (m_i, o_i)_{i=1..5}\} \subseteq \prec_\varepsilon^+ .$$

In other words, the rates of m are at most 2ε-smaller than the corresponding rates of n, or larger; and m has at most ε-smaller rates than the corresponding ones of o, but not larger.

The next lemma derives from Theorem 1 and it proves that the concept of ε-behavioral order generalizes the concept of stochastic bisimulation.

Lemma 5. *Any bisimulation relation is an ε-behavioral order for any rational $\varepsilon \geq 0$. Moreover, if $m \sim n$ then $m \prec_0 n$ and $n \prec_0 m$.*

The next theorem generalizes the theorems 7 and 8 for behavioral orders. In this new context the ε-semantics is the key.

Theorem 9 (Logical characterization of $\prec_\varepsilon$)**.** *For arbitrary rational $\varepsilon \geq 0$,*

$$[\text{for any } \phi \in \mathcal{L}^+, n \models \phi \text{ implies } m \models_\varepsilon \phi] \text{ iff } m \prec_\varepsilon n.$$

Proof: ($\Longrightarrow$) Let $\triangleleft : M \times M$ be defined by $m \triangleleft n$ iff [for any $\phi \in \mathcal{L}^+, n \models \phi$ implies $m \models_\varepsilon \phi$]. We prove that $\triangleleft$ is an ε-behavioral order.
It is trivial to verify that $\triangleleft$ is closed to bisimulation as the ε-semantics preserves bisimulation.
Let $C \in G_\mathcal{M}$ be an arbitrary set. From Corollary 1 we know that there exists $\phi \in \mathcal{L}^+$ such that $C = [\![\phi]\!]$. From the definition of $\triangleleft$ and Corollary 2, $C^\triangleleft = [\![\langle\phi\rangle_\varepsilon]\!]$.
Suppose that $\theta(n)([\![\phi]\!]) = p$. Then, $n \models L_p\phi$, implying $m \models_\varepsilon L_p\phi$ which is equivalent to $m \models L_{p-\varepsilon}\langle\phi\rangle_\varepsilon$. Consequently, $\theta(m)([\![\langle\phi\rangle_\varepsilon]\!]) \geq p - \varepsilon$ implying $\theta(n)(C) \leq \theta(m)(C^\triangleleft) + \varepsilon$. Hence $\triangleleft$ is an ε-behavioral order.
($\Longleftarrow$) We prove it by induction on ϕ.
The cases $\phi = \top$, $\phi = \phi_1 \wedge \phi_2$ and $\phi = \phi_1 \vee \phi_2$ are trivial.
The case $\phi = L_r\psi$**:** $n \models L_r\psi$ implies $\theta(n)([\![\psi]\!]) \geq r$. Because $m \prec_\varepsilon n$, we have that $\theta(n)([\![\psi]\!]) \leq \theta(m)([\![\langle\psi\rangle_\varepsilon]\!]) + \varepsilon$ and we obtain further that $\theta(m)([\![\langle\phi\rangle_\varepsilon]\!]) \geq r - \varepsilon$. Consequently, $m \models L_{r-\varepsilon}\langle\phi\rangle_\varepsilon$, i.e., $m \models_\varepsilon L_r\psi$. ■

The previous theorem can be further generalized to comprise also the negative formulas. In order to do that, because there is an asymmetry between the behavior of the positive and negative formulas in the ε-semantics, we will need an extra encoding that we define below. This encodings assumes that the formulas are in disjunctive normal forms (when $L_r\phi$ are considered atoms).

$$|\phi|_\varepsilon = \begin{cases} \top & \text{if } \phi = \top \\ \bot & \text{if } \phi = \bot \\ L_r|\psi|_\varepsilon & \text{if } \phi = L_r\psi \\ \neg L_{r+\varepsilon}|\psi|_\varepsilon & \text{if } \phi = \neg L_r\psi \\ |\psi_1|_\varepsilon \wedge |\psi_2|_\varepsilon & \text{if } \phi = \psi_1 \wedge \psi_2 \\ |\psi_1|_\varepsilon \vee |\psi_2|_\varepsilon & \text{if } \phi = \psi_1 \vee \psi_2 \end{cases}$$

With this encoding we can state the generalization of the previous theorem.

Theorem 10 (Logical characterization of $\prec_\varepsilon^+$)**.** *For arbitrary rational $\varepsilon \geq 0$,*

$$[\text{for any } \phi \in \mathcal{L}, n \models \phi \text{ implies } m \models_\varepsilon |\phi|_\varepsilon] \text{ iff } m \prec_\varepsilon^+ n.$$

Proof: ($\Longrightarrow$) Let $\triangleleft : M \times M$ be defined by $m \triangleleft n$ iff [for any $\phi \in \mathcal{L}, n \models \phi$ implies $m \models_\varepsilon |\phi|_\varepsilon$]. We prove that $\triangleleft$ is an extended ε-behavioral order.
It is trivial to verify that $\triangleleft$ is closed to bisimulation as the ε-semantics preserves bisimulation.
Let $C \in \overline{G_\mathcal{M}}$ be an arbitrary set. From Corollary 1 we know that there exists $\phi \in \mathcal{L}$ such that $C = [\![\phi]\!]$. From the definition of $\triangleleft$ and Corollary 2, $C^\triangleleft = [\![\langle|\phi|_\varepsilon\rangle_\varepsilon]\!]$.
Suppose that $\theta(n)([\![\phi]\!]) = p$. Then on one hand, $n \models L_p\phi$, implying $m \models_\varepsilon L_p\langle\phi\rangle$ which is equivalent to $m \models L_{p-\varepsilon}\langle|\phi|_\varepsilon\rangle_\varepsilon$. Consequently, $\theta(m)([\![\langle|\phi|_\varepsilon\rangle_\varepsilon]\!]) \geq p - \varepsilon$ implying $\theta(n)(C) - \theta(m)(C^\triangleleft) \leq \varepsilon$.
On the other hand, from $\theta(n)([\![\phi]\!]) = p$ we obtain that for any $\delta > 0$, $n \models \neg L_{p+\delta}\phi$ implying $m \models_\varepsilon \neg L_{p+\delta+\varepsilon}\langle\phi\rangle$ which is equivalent to $m \models \neg L_{p+\delta}\langle|\phi|_\varepsilon\rangle_\varepsilon$. Hence, for each $\delta > 0$, $\theta(m)([\![\langle|\phi|_\varepsilon\rangle_\varepsilon]\!]) < p + \delta$ which implies $\theta(n)([\![\phi]\!]) - \theta(m)([\![\langle|\phi|_\varepsilon\rangle_\varepsilon]\!]) \geq 0$.
Hence $\triangleleft$ is an extended ε-behavioral order.
($\Longleftarrow$) We prove it by induction on ϕ.
The cases $\phi = \top$, $\phi = \phi_1 \wedge \phi_2$ and $\phi = \phi_1 \vee \phi_2$ are trivial.
The case $\phi = L_r\psi$**:** $n \models L_r\psi$ implies $\theta(n)([\![\psi]\!]) \geq r$. Because $m \prec_\varepsilon^+ n$, we have that $\theta(n)([\![\psi]\!]) \leq \theta(m)([\![\langle|\psi|_\varepsilon\rangle_\varepsilon]\!]) + \varepsilon$ and we obtain further that $\theta(m)([\![\langle|\phi|_\varepsilon\rangle_\varepsilon]\!]) \geq r - \varepsilon$. Consequently, $m \models L_{r-\varepsilon}\langle|\phi|_\varepsilon\rangle_\varepsilon$, i.e., $m \models_\varepsilon |L_r\psi|_\varepsilon$.
The case $\phi = \neg L_r\psi$**:** $n \models \neg L_r\psi$ implies $\theta(n)([\![\psi]\!]) < r$. Because

$m \prec^{+}_{\varepsilon} n$, we have that $\theta(n)([\![\psi]\!]) - \theta(m)([\![\langle|\psi|_{\varepsilon}\rangle_{\varepsilon}]\!]) \geq 0$ and we obtain further that $\theta(m)([\![\langle|\psi|_{\varepsilon}\rangle_{\varepsilon}]\!]) < r$. Consequently, $m \models \neg L_r\langle|\psi|_{\varepsilon}\rangle_{\varepsilon}$, i.e., $m \models_{\varepsilon} |\neg L_r\psi|_{\varepsilon}$. ∎

VII. The pseudometrizable space of processes

In what follows we use $\prec_{\varepsilon}$ to define a canonic distance between CMPs that resemble (and generalize) the well known point-wise distances for particular types of CMPs such as Markov chains.

The next result guarantees that between any two systems there is a $\prec_{\varepsilon}$ relation for an ε that is big enough.

Lemma 6. *For any couple of CMPs m and n there exists a positive rational ε such that $m \prec_{\varepsilon} n$.*

Proof: It is sufficient to notice that for any $C \in G_{\mathcal{M}}$ and any $\varepsilon \geq 0$ we have that

$$\theta(n)(C) - \theta(m)(C^{\prec_{\varepsilon}}) \leq \theta(n)([\![\top]\!]_{\varepsilon}) = \theta(n)([\![\top]\!]).$$

Consequently, in the worst case we can take $\varepsilon \in [\theta(n)([\![\top]\!]), \infty) \cap \mathbb{Q}^{+}$. ∎

The previous lemma allows us to define a function

$$d : M \times M \to \mathbb{R}^{+} \text{ by } d(m,n) = inf\{\varepsilon \mid m \prec_{\varepsilon} n \text{ and } n \prec_{\varepsilon} m\}.$$

As stated in the next theorem, d is a pseudometric on M that measures how different two systems are from the point of view of their behavior. The distance between two systems is 0 iff the systems are bisimilar.

Theorem 11 (Pseudometric)**.** *The function $d : M \times M \to \mathbb{R}^{+}$ defined before is a pseudometric on M which characterizes stochastic bisimulation, i.e.,*

$$d(m,n) = 0 \textit{ iff } m \sim n.$$

Proof: We prove for the beginning that d is a pseudometric, i.e., that is symmetric and satisfies the triangle inequality. The symmetry is an immediate consequence of the definition. To prove the triangle inequality, we need to prove that for arbitrary $\phi \in \mathcal{L}^{+}$, if we have

$$\theta(m)([\![\phi]\!]) - \theta(n)([\![\langle\phi\rangle_{\varepsilon}]\!]) \leq \varepsilon \quad (1)$$
$$\theta(n)([\![\phi]\!]) - \theta(m)([\![\langle\phi\rangle_{\varepsilon}]\!]) \leq \varepsilon \quad (2)$$
$$\theta(p)([\![\phi]\!]) - \theta(m)([\![\langle\phi\rangle_{\varepsilon'}]\!]) \leq \varepsilon' \quad (3)$$
$$\theta(m)([\![\phi]\!]) - \theta(p)([\![\langle\phi\rangle_{\varepsilon'}]\!]) \leq \varepsilon' \quad (4)$$

then

$$\theta(n)([\![\phi]\!]) - \theta(p)([\![\langle\phi\rangle_{\varepsilon+\varepsilon'}]\!]) \leq \varepsilon + \varepsilon' \quad (5)$$
$$\theta(p)([\![\phi]\!]) - \theta(n)([\![\langle\phi\rangle_{\varepsilon+\varepsilon'}]\!]) \leq \varepsilon + \varepsilon' \quad (6)$$

To prove (5), we sum (2) with the instance of (3) for $\langle\phi\rangle_{\varepsilon}$ instead of ϕ. And to prove (6), we sum (1) with the instance of (4) for $\langle\phi\rangle_{\varepsilon}$ instead of ϕ.

It remains to prove that $d(m,n) = 0$ iff $m \sim n$.
($\Longleftarrow$) This direction derives from Lemma 5 where we proved that any bisimulation relation is a 0-behavioral order.

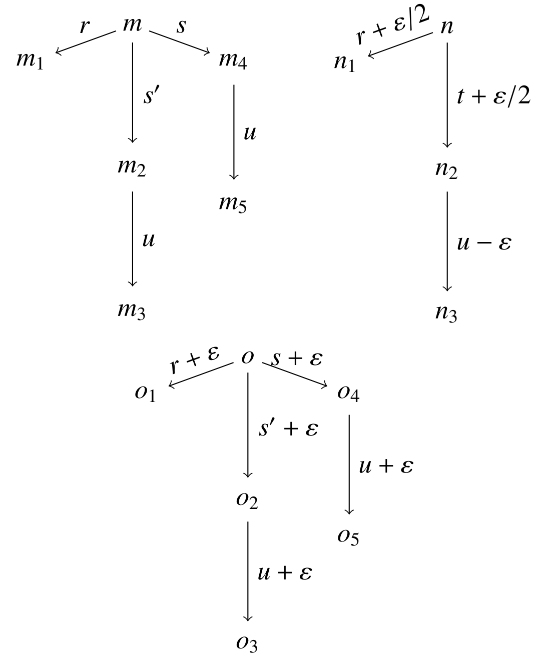

Fig. 3. Distances between Markovian processes.

($\Longrightarrow$) If $d(m,n) = 0$, then for any rational $\varepsilon > 0$ we have that $m \prec_{\varepsilon} n$ and $n \prec_{\varepsilon} m$. So, using Theorem 9, for any rational $\varepsilon > 0$ and any $\phi \in \mathcal{L}^{+}$, $[m \models \phi$ implies $n \models_{\varepsilon} \phi]$ and $[n \models \phi$ implies $m \models_{e} \phi]$. These can be rewritten as $[n \models \phi$ implies that for any $\varepsilon > 0$, $m \models_{\varepsilon} \phi]$ and $[m \models \phi$ implies that for any $\varepsilon > 0$, $n \models_{\varepsilon} \phi]$. Using Lemma 3 we obtain that [for any $\phi \in \mathcal{L}^{+}$, $m \models \phi$ iff $n \models \phi$] which is equivalent to $m \sim n$. ∎

To conclude this section and understand the significance of this distance, we shall take a look to the following example.

Example 4. *Consider the CMPs described in Figure 3. We notice that the processes with initial states m and o are quite similar as structure and rate values; the second one has all the transitions ε-bigger than the first one. So a first guess will be that $d(m,o) = \varepsilon$. But this is not the case because the rate of exiting the state m is*

$$\theta(m)(\{m_1, m_2, m_3, m_4, m_5\}) = r + s + s',$$

which is 3ε smaller than the rate of exiting the state o

$$\theta(o)(\{o_1, o_2, o_3, o_4, o_5\}) = r + s + s' + 3\varepsilon.$$

Consequently, $d(m,o) = 3\varepsilon$.

Consider now the CMPs with m and n as initial states and suppose, as before, that $s + s' = t$. We should notice this time that not all the transitions of the first CMP are bigger than the transitions of the second. However, every pair of transitions do not differ with more than ε. And since there are paired transitions that differ with exactly ε value, we obtain that $d(m,n) = \varepsilon$.

VIII. Conclusions

In this paper we have introduced a parametric metatheory for Continuous Markovian Logic. The parameter ε of the metatheory encodes an observation error that might appear when we analyze a stochastic system. We define an ε-semantics and an axiomatized ε-proof system and we show that the ε-provability relation is sound and complete with respect to the ε-satisfiability relation. This entire logical framework also allows us to transfer metaproperties between various ε-levels of the metatheory. We prove a series of results regarding the connection between ε-satisfiability and $\varepsilon+\varepsilon'$-satisfiability and a parametrized deduction theorem that combines ε-provability and $\varepsilon+\varepsilon'$-provability results.

This classic metalogical framework allows us to give an uniform treatment to all logical properties, including the ones involving negative or logical implication, while avoiding unorthodox logical constructs as the real-valued logics. The framework also supports us in identifying two canonical behavioural orders that extend stochastic bisimulation and organize the space of CMPs. These bisimulation orders are the cornerstones in the definition of a pseudometric on CMPs that measure the behavioral similarity of processes.

The metalogical framework introduced in this paper can be particularized to more specific Markovian models such as the discrete or continuous-time Markov chains. Moreover, the entire development can be adapted to specialize on the probabilistic cases, as the mathematical structure that supports the definition of CMPs is similar to the one that supports the definition of labeled Markov processes in the form of [Pan09].

This paper opens a series of interesting research questions regarding the relation between ε-satisfiability, ε-provability and metric semantics. There are many open questions related to the possibility of defining a pseudometric over the class of logical formulas that shall measure ε-provability; for instance such that the distance between ϕ and ψ is 0 iff ϕ and ψ are logical equivalent. On this direction we expect to be able to prove a version of metric completeness that relates the pseudometric space of CMPs to the pseudometric space of logical formulas. The first two authors in collaboration with Prakash Panangaden have already obtained a series of results in this direction [LMP12a], [LMP12b]; but these results do not involve the parametric metatheory yet. And the hope is that the new metatheoretical perspective introduced in this paper will eventually solve some of the open problems that resisted to the other approaches.

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